MATH 890 – FOURIER ANALYSIS – F13

HW 4. More on reproducing formulas

1. Let $f \in L^2(R)$ with $\operatorname{supp} \hat{f} \subset [-\pi, \pi]$. Show that, for $|x| < N/2$, the *truncation error*

$$T_N f(x) = f(x) - \sum_{|k| \leq N} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)}$$

satisfies the estimate

$$|T_N f(x)| \leq C \|f\|_{L^2} N^{-1/2}.$$

(Hint: Cauchy-Schwarz.)

2. Show that if $f \in L^2(R)$ and $\operatorname{supp} \hat{f} \subset [-\pi, \pi]$, then

$$\|f'\|_{L^2} \leq C \|f\|_{L^2}.$$

3. Let $f \in L^2(R)$, $\operatorname{supp} \hat{f} \subset [-\pi, \pi]$, and let $\{\epsilon_k\}$ be a sequence of (small) positive numbers. Show that the *jitter error*

$$Jf(x) = f(x) - \sum_{k \in \mathbf{Z}} f(k + \epsilon_k) \frac{\sin \pi(x-k)}{\pi(x-k)}$$

satisfies the estimate

$$\|Jf\|_{L^2} \leq C \|f\|_{L^2} (\sup \epsilon_k)^{1/2}.$$

(Hint: estimate the norm using samples. Write $f(k + \epsilon_k) - f(k)$ as a n integral and use problem 2.)

4. Give the necessary details to show that the Calderón reproducing formula converges in L^2 and not just in $L^2 \cap L^1$.

5. Recall (or see for example Folland's *Real analysis*, pp. 242-244) that if $\psi \in \mathcal{S}$, and $f \in L^2$, then $f * \psi_t$ converges a.e. to $(\int \psi(u) du) f$.

a) Show that if $\varphi \in \mathcal{S}$ satisfies the hypothesis in the theorem about Calderón's reproducing formula, then there exists a function $\psi \in \mathcal{S}$ such that $\int \psi(u) du = 1$ and

$$-t \frac{d}{dt} (\psi_t * f) = \varphi_t * \varphi_t * f.$$

(Hint: take the Fourier transform and solve the simple ODE.)

b) Show that the Calderón reproducing formula with $\varphi \in S$ and $f \in L^2$ converges a.e. (Hint: use part a) and the fact recalled at the beginning of the exercise.)