Homewrok 5 - Pb 5

$$\rho(f,g) = \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\, d\mu(x)$$

ρ IS A METRIC :

i) $\rho(f,g) = 0 \iff \dfrac{|f(x)-g(x)|}{1+|f(x)-g(x)|} = 0$ a.e x

$\iff f(x) = g(x)$ a.e x.

ii) $\rho(f,g) = \rho(g,f)$ (CLEAR) ✓

iii) NOTE THAT THE FUNCTION

$F(x) = \dfrac{x}{1+x}$ IS INCREASING ON $\mathbb{R}^+$

BECAUSE $F'(x) = \dfrac{1}{(1+x)^2} > 0$

THEN, SINCE $|f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|$

WE HAVE

$$\frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} = F(|f(x)-g(x)|) \leq F(|f(x)-h(x)|+|h(x)-g(x)|)$$

$$= \frac{|f(x)-h(x)|+|h(x)-g(x)|}{1+\underbrace{|f(x)-h(x)|}_{\geq 0}+\underbrace{|h(x)-g(x)|}_{\geq 0}} \leq \frac{|f(x)-h(x)|}{1+|f(x)-h(x)|} + \frac{|h(x)-g(x)|}{1+|h(x)-g(x)|}$$

NOW. INTEGRATE TO OBTAIN THE TRIANGLE INEQUALITY FOR ρ. —

$\rho(f_n, f) \longrightarrow 0 \iff f_n \longrightarrow f$ IN MEASURE :

$\Rightarrow)$ LET $E_{n,\delta} = \{ x : |f_n(x) - f(x)| > \delta \}$

WE WANT TO SHOW THAT FOR $\delta > 0$ <u>FIXED</u>
$\mu(E_{n,\delta}) \longrightarrow 0$ AS $n \longrightarrow \infty$. BUT,

$$\rho(f_n, f) = \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu \geq \int_{E_{n,\delta}} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu$$

$$\underset{\uparrow}{\geq} \int_{E_{n,\delta}} \frac{\delta}{1+\delta} \, d\mu = \frac{\delta}{1+\delta} \mu(E_{n,\delta})$$

(SINCE F IS $\uparrow$)

$$\Longrightarrow \mu(E_{n,\delta}) \leq \frac{1+\delta}{\delta} \rho(f_n, f) \longrightarrow 0 \quad \text{IF } n \longrightarrow \infty$$

$$\Leftarrow) \quad \rho(f_n, f) \leq \int_{E_{n,\delta}} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu + \int_{E^c_{n,\delta}} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu$$

$$\leq \int_{E_{n,\delta}} 1 \, d\mu + \int_{E^c_{n,\delta}} \frac{\delta}{1+\delta} \, d\mu$$

$$\leq \mu(E_{n,\delta}) + \frac{\delta}{1+\delta} \underbrace{\mu(X)}_{<\infty}$$

GIVEN NOW $\varepsilon > 0$, PICK $\delta > 0$ SO THAT
$\delta/_{1+\delta} \, \mu(X) < \varepsilon/2$ AND THEN n LARGE SO THAT
$\mu(E_{n,\delta}) < \varepsilon/2$, TO OBTAIN
$\rho(f_n, f) < \varepsilon$ IF n IS LARGE. —