MATH 766 – Spring 2012

PRACTICE PROBLEMS FOR THE FINAL EXAM

The exam will be based on the material in the book form Sections 8.1, 8.2, 10.1, 10.2, 10.3, 10.4, 10.5, 10.6, 11.1, 11.2 11.3, 11.4, 11.5 and 11.6 that we covered in class. (Note that some of the results in 8.3, 8.4, 9.1, 9.2, 9.3, and 9.4 are particular case of the results for metric spaces, so you may want to read them too). If in doubt please ask me.

- Of course, you need to know all the definitions of the concepts introduced and be able to state them.
- Study the proofs of the results presented in the lectures. Becoming acquainted with the proofs can help you prove similar related results.
- Prove and/or complete the properties and simple facts left as exercises in class.
- Review your homework and make sure you know how to do the all the problems assigned.
- Do the following practice problems.
- **You should expect about 2-3 problems from the material we covered in the two midterms and about 3-4 problems from the material in Chapter 11.**

Problem 1.
Let E be a nonempty closed set and K be a nonempty compact set of $\mathbb{R}^n$. Show that there exist $c > 0$ such that $d(e, k) > c$ for all $e \in E$ and all $k \in K$.

Problem 2.
Let X be a complete metric space with metric ρ.

a) Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of point in X with the property that $\rho(x_{n+1}, x_n) \leq c\,\rho(x_n, x_{n-1})$ for some fixed $0 < c < 1$ and all n in $\mathbb{N}$. Show that the sequence is convergent. (Hint: $\sum_{j=n}^{m} c^j \leq \epsilon$ in n, m are large.)

b) Let $f : X \to X$ be a function with the property that $\rho(f(x), f(y)) \leq c\,\rho(x, y)$ for some fixed $0 < c < 1$ and all x, y in X. Fix a point x_0 in X and define $\{x_n\}_{n=0}^{\infty}$ where $x_1 = f(x_0)$, $x_2 = f(x_1)$, ... (i.e. $x_{n+1} = f(x_n)$). Show that this sequence converges to some point x in X.

c) Show that the limit of the sequence in b) is a fixed point of f, i.e., it solves the equation $x = f(x)$. (Hint: f is obviously continuous.)

Problem 3.
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function whose derivative with respect to the first variable, f_x, exists and is continuous on $\mathbb{R}^2$.

a) Show that

$$lim_{h\to 0} \max_{a\le y\le b}\left(\frac{f(x+h,y)-f(x,y)}{h}-f_x(x,y)\right)=0$$

b) Show that

$$\frac{d}{dx}\int_a^b f(x,y)\,dy=\int_a^b f_x(x,y)\,dy.$$

Problem 4.
Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by $f(x,y)=x^3(x^2+y^2)^{-1}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$.

a) Show that the partial derivatives f_x and f_y exist at $(0,0)$.

b) Show that f is not differentiable at $(0,0)$. (Hint: Consider in the definition of differentiability an increment $h=(t,t)$.)

Problem 5.
Let $f,g:\mathbb{R}\to\mathbb{R}$ be continuously differentiable functions such that $f(0)=g(0)=0$ and $f'(0)\neq 0$. Show that there exist and open set V containing $(0,0)$ in $\mathbb{R}^2$ and a continuously differentiable function $h:V\to\mathbb{R}$ such that $h(z,w)=0$ if an only if $(z,w)=(f(x),g(x))$ for x near 0. (Hint: Consider $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y)=(f(x),g(x)+y)$, use the Inverse Function Theorem, and discover the function h needed.)

Problem 6. Show that there exist $\delta>0$ and a continuously differentiable function f on the interval $(-\delta,\delta)$ such that $xf^3(x)+f(x)=1$ for all x in $(-\delta,\delta)$. Compute $f'(0)$.

Problem 7. The surfaces

$$x^2+y^2+z^2=1$$

and

$$xy+z=1/2$$

intersect in a curve C. Show that for each point $p_0=(x_0,y_0,z_0)$ on C there exist and open interval I in $\mathbb{R}$ and a function $\Phi_{p_0}:I\to\mathbb{R}^3$ such that $p_0\in\Phi_{p_0}(I)\subset C$. Moreover, show that Φ_{p_0} is one-one.