MATH 766 – Spring 12

Additional practice problems for Exam 2

- The exam will be based on the material in the book form Sections 10.1, 10.2, 10.3, 10.4, 10.5, and 10.6 that we covered in class. (Note that some of the results in 8.3, 8.4, 9.3, 9.4 are particular case of the results for metric spaces, so you may want to read them too). If in doubt please ask me.
- Of course, you need to know all the definitions of the concepts introduced and be able to state them.
- Study the proofs of the results presented in the lectures. Becoming acquainted with the proofs can help you prove similar related results.
- Prove and/or complete the properties and simple facts left as exercises in class.
- Review your homework and make sure you know how to do the all the problems assigned.
- Do the following practice problems. Warning: These are just practice problems to help you further understand some of the concepts we covered so far. This is not a sample test.

Further problems
Problem 1.

Let X be a metric space. True or False? (If True just say so; if False give a counterexample.)

a) Every compact set in X is closed and bounded.

b) Every closed and bounded set in X is compact.

Problem 2.
Let $\mathbb{Q}^2 = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\}$ and consider it as a metric space with the usual Euclidean metric. Does $\mathbb{Q}^2$ satisfy the Bolzano-Weirestrass Property? Justify your answer. Is $\mathbb{Q}^2$ connected? Justify your answer.

Problem 3.
Let E be subset of $\mathbb{R}^2$ given by

$$E = (B_1((0,0)) \setminus \{(0,0)\}) \cup \{(1 + 1/k, 0) : k \in \mathbb{N}\}$$

a) Compute $E^o, \bar{E}, \partial E$.

b) Which of the following are relatively open and/or relatively closed sets in E? Justify your answers.

- $B_1((0,0)) \setminus \{(0,0)\}$
- $(B_1((0,0)) \setminus \{(0,0)) \cup \{(1,0)\}$
- $(B_1((0,0)) \setminus \{(0,0)) \cup \{(3/2,0)\}$

Problem 4.
Do problems 10.3.11 in the book.

Problem 5.
Let E be a bounded set in $\mathbb{R}^n$. Show that every sequence of points in E has a subsequence convergent to a point in $\bar{E}$.

Problem 6.
Do problem 10.4.7 in the book.

Problem 7.
Do problem 10.2.7 in the book.

Problem 8.
Let E be a set in $\mathbb{R}^n$ and let $f : E \to \mathbb{R}$ be a continuous function at $a \in E$. Show that if $f(a) > 0$ then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in B_\delta(a) \cap E$.

Problem 9.
Let $X = C[a, b]$ be the metric space of real valued continuous functions on the interval $[a, b]$ with the usual metric $\rho(f, g) = \|f - g\|$, where $\|f\| = \sup_{x\in[a,b]} |f(x)|$. Show that the function $E : X \to \mathbb{R}$ given by $E(f) = f(a)$ is continuous on X.

Problem 10.
Let $\{K_\alpha\}_{\alpha\in A}$ be a collection of compact sets in $\mathbb{R}^n$ with the property that any finite sub-collection of them has nonempty intersection. Shown that $\cap_{\alpha\in A} K_\alpha \neq \emptyset$. (Hint: Suppose that for some α_0, $K_{\alpha_0} \cap (\cap_{\alpha\neq\alpha_0} K_\alpha) = \emptyset$, then $K_{\alpha_0} \subset \cup_{\alpha\neq\alpha_0} K_\alpha^c$. Show that this leads to a contradiction.)

Problem 11.
Show that if $\{K_j\}_{j\in\mathbb{N}}$ is a sequence of nonempty compact sets such that $K_{j+1} \subset K_j$ then $\cap_{j\in\mathbb{N}} K_j \neq \emptyset$.

Problem 12.
Let X be a metric space and let $f : X \to \mathbb{R}$ be a *uniformly continuous* function. Show that if $\{x_n\}$ is a Cauchy sequence in X then $\{f(x_n)\}$ converges in $\mathbb{R}$. Is this true if *uniformly continuous* is replaced by *continuous*? Prove it or give a counterexample.