Bonus work (worth 20 points)

1) For E_1 and E_2, two sets in $\mathbb{R}^n$, define

$$E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}.$$

a) Prove that if E_1 and E_2 are compact then $E_1 + E_2$ is also compact.

b) Give an example of a closed set E in $\mathbf{R}$ such that $E + \mathbb{N}$ is not closed (here $\mathbb{N}$ is the set of natural numbers).

2) Show that if $\sum_{n=1}^{\infty} f_n$ converges pointwise to a continuous function f on $[0, 1]$ and every f_n is continuous and non-negative on $[0, 1]$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to f.

(Hint: For $\epsilon > 0$, consider the sets $K_N = \{x : f(x) - \sum_{n=1}^{N} f_n(x) \geq \epsilon\}$, and show that their intersection has to be empty. Then use some properties of compact sets.)

3) Let $C[0, 1]$ be the set of all real-valued continuous functions on $[0, 1]$. Let $\psi \in C[0, 1]$ and define

$$\rho_\psi(f, g) = \int_0^1 \psi(x) \, |f(x) - g(x)| \, dx.$$

a) Show that if $\psi(x) > 0$ for all $x \in [0, 1]$, then ρ_ψ is a metric in $C[0, 1]$.

b) Show that if $\psi(x) = 0$ for $0 \leq x \leq 1/2$ and $\psi(x) = x - 1/2$ for $1/2 \leq x \leq 1$, then ρ_ψ is not a metric in $C[0, 1]$.

4) Let X be a metric space with metric ρ, and let E be a closed subset of X. Show that the function $f : X \to [0, \infty)$ defined by

$$f(x) = \inf\{\rho(x, y) : y \in E\}$$

is continuous, and that $f(x) = 0$ if an only if $x \in E$.

5) Let $f : U \to V$ be a continuously differentiable function between two open sets in $\mathbb{R}^n$. Suppose that the Jacobian determinant of f is never zero on U, that $f^{-1}(K)$ is compact for any compact set $K \subset V$, and that V is connected. Show that $f(U) = V$.