Homework 7

Math 766
Spring 2012

10.6.6 Suppose that H is a nonempty compact subset of X and that Y is a Euclidean space.

a) If $f : H \to Y$ is continuous, prove that

$$||f||_H := \sup_{x \in H} ||f(x)||_Y$$

is finite and there exists $x_0 \in H$ such that $||f(x_0)||_Y = ||f||_H$.
Proof: Since f is continuous on a compact set H, f is uniformly continuous on H. Fix $\varepsilon > 0$, and there exists $\delta > 0$ such that $d_H(x,y) < \delta$ implies that $d_Y(f(x), f(y)) < \varepsilon$. Since Y is a Euclidean space $d_Y(y_1, y_2) = ||y_1 - y_2||_Y$ where $||\cdot||_Y$ is a Euclidean norm. Consider the function $g : H \to \mathbb{R}$ defined $g(x) = ||f(x)||_Y$. Then for $d_H(x,y) < \delta$

$$|g(x) - g(y)| = |\,||f(x)||_Y - ||f(y)||_Y| \leq ||f(x) - f(y)||_Y = d_Y(f(x), f(y)) < \varepsilon.$$

Therefore $g : H \to \mathbb{R}$ is continuous, and by the extreme value theorem

$$||f||_H = \sup_{x \in H} g(x)$$

is finite and there exists $x_0 \in H$ such that $||f||_H = f(x_0)$. □

10.6.8 Suppose $E \subset X$ and that $f : E \to Y$.

a) If f is uniformly continuous on E and $x_n \in E$ is Cauchy in X, prove that $f(x_n)$ is Cauchy in Y.
Proof: Let $\varepsilon > 0$ be arbitrary. Since $f : E \to Y$ is uniformly continuous, there exists $\delta > 0$ such that

$$d_x(x_1, x_2) < \delta, \;\; x_1, x_2 \in E \Longrightarrow d_y(f(x_1), f(x_2)) < \varepsilon$$

where d_X and d_Y be the metric on X and Y respectively. Since $x_n \in X$ is Cauchy, there exists $N \in \mathbb{N}$ such that

$$m, n > N \Longrightarrow d_X(x_m, x_n) < \delta.$$

Then for $m, n > N$ it follows that

$$d_Y(f(x_m, x_n)) < \varepsilon.$$

Therefore $f(x_n)$ is Cauchy in Y. □

b) Suppose that D is a dense subspace of X. If Y is complete and $f : D \to Y$ is uniformly continuous on D, prove that $f$has a continuous extension to X.

Proof: Fix $x \in X$, and there exists $x_n \in D$ such that $x_n \to x$. Since x_n is convergent in X, it follows that x_n is Cauchy in X. By part **a)** it follows that $f(x_n)$ is Cauchy in Y. Since Y is complete, there exists $y_x \in Y$ such that $f(x_n) \to y_x$ in Y. So given $x \in X$, define $g(x) = y_x$. By the uniqueness of limits, $g : X \to Y$ is well-defined.

$\underline{g|_D = f}$: For $x \in D$, take $x_n = x$ for all n. Then $x_n \to x$ in X and hence

$$g(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x) = f(x).$$

Therefore $g(x) = f(x)$ for all $x \in D$.

$\underline{g \text{ is continuous}}$: Let $\varepsilon > 0$ and $x_0 \in X$. Since f is uniformly continuous on D, there exists $\delta > 0$ such that

$$d_X(x,y) < \delta,\ \ x,y \in D \Longrightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Fix $x \in X$ such that $d_X(x_0,x) < \delta/3$. There exist $x_n^0, x_n \in D$ such that $x_n^0 \to x_0$ and $x_n \to x$ in X. Then there exists $N \in \mathbb{N}$ such that $n \geq N_1$ implies that $d_X(x_n^0, x_0) < \delta/3$ and $d_X(x_n,x) < \delta/3$. Then for $n \geq N_1$

$$d_X(x_n^0, x_n) \leq d_X(x_n^0, x_0) + d_X(x_0,x) + d_X(x,x_n) < \delta.$$

By the definition of g, we have $f(x_n^0) \to g(x_0)$ and $f(x_n) \to g(x)$ in X. So there exists N_2 such that $n \geq N_2$ implies that $d_Y(g(x_0), f(x_n^0)) < \varepsilon$ and $d_Y(g(x), f(x_n)) < \varepsilon$. Now fix $n_0 > \max(N_1,N_2)$, and it follows that

$$\begin{aligned} d_Y(g(x_0), g(x)) &\leq d_Y(g(x_0), g(x_n^0)) + d_Y(g(x_n^0), g(x_n)) + d_Y(g(x_n), g(x)) \\ &= d_Y(g(x_0), f(x_n^0)) + d_Y(f(x_n^0), f(x_n)) + d_Y(f(x_n), g(x)) \\ &\leq 3\varepsilon. \end{aligned}$$

Therefore g is continuous on X. □