Homework 4

Math 766
Spring 2012

8.2.11 Fix $T \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$. Set

$$M_1 := \sup_{||x||=1} ||T(x)||$$

$$M_2 := \int \{C > 0 : ||T(x)|| \leq C||x|| \text{ for all } x \in \mathbb{R}^n\}.$$

a) Prove that $M_1 \leq ||T||$.
Proof: Let $x \in \mathbb{R}^n$ and note that $x/||x|| = 1$. Then

$$M_1 = \sup_{||x||=1} ||T(x)|| = \sup_{||x||=1} \frac{||T(x)||}{||x||} \leq \sup_{||x||\neq 0} \frac{||T(x)||}{||x||} = ||T||.$$

□

b) Using the linear property of T, prove that if $x \neq 0$, then

$$\frac{||T(x)||}{||x||} \leq M_1.$$

Proof: Suppose $x_0 \neq 0$, and define $y = x_0/||x_0||$ which satisfies $||y|| = 1$. Then using the linearity of T

$$\frac{||T(x_0)||}{||x_0||} = ||T(x_0/||x_0||)|| = ||T(y)|| \leq \sup_{||x||=1} ||T(x)|| = M_1.$$

□

c) Prove that $M_1 = M_2 = ||T||$.
Proof: Combining parts **a)** and **b)**, for all $x \neq 0$,

$$\frac{||T(x)||}{||x||} \leq M_1 \leq ||T||.$$

Then taking the supremum over $x \neq 0$, it follows that $||T|| \leq M_1 \leq ||T||$. That is $M_1 = ||T||$. If $M_1 = 0$, then for all $x \neq 0$

$$0 \leq ||T(x)|| \leq M_1||x|| = 0$$

and hence $T = 0$. Then trivially $M_1 = M_2 = ||T||$. If $M_1 \neq 0$, then $M_1 > 0$. It follows that $M_1 \in \{C > 0 : ||T(x)|| \leq C||x|| \text{ for all } x \in \mathbb{R}^n\}$ since $T(0) = 0 \leq M_1||0|| = 0$ and by part **b)** for $x \neq 0$

$$||T(x)|| \leq M_1||x||.$$

Therefore $M_1 \geq M_2$. On the other hand, for any $C > 0$ such that $||T(x)|| \leq C||x||$ for all $x \in \mathbb{R}^n$,

$$M_1 = \sup_{||x||=1} ||T(x)|| \leq \sup_{||x||=1} C||x|| = C.$$

So taking the inf for all such C, it follows that $M_1 \leq M_2$ as well. Therefore $M_1 = M_2 = ||T||$.$\square$