Summary of the fundamental theorems of vector calculus

Math 123

1 Fundamental theorem for path integrals.

1. (Theorem 6.1.1 in the book) Let $U \subset \mathbb{R}^n$ be an open connected set and let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function whose gradient is continuous on U. If C is a continuous smooth path lying on U that joins a point $\mathbf{a}$ to another point $\mathbf{b}$, then

$$\int_C \vec{\nabla} f \cdot d\mathbf{x} = f(\mathbf{b}) - f(\mathbf{a}).$$

2. We also have that for a continuous vector field $\mathbf{F} : U \to \mathbb{R}^n$, $\mathbf{F}$ is **path independent on** U if and only if there exists a scalar-valued function $f : U \to \mathbb{R}$ such that $\vec{\nabla} f = \mathbf{F}$ and if and only if $\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$ for all closed paths C that lie in U.

3. (Theorem 6.1.5 in the book) Let U be a **simply connected open set** in $\mathbb{R}^n$ and let $\mathbf{F} : U \to \mathbb{R}^n$ be a **vector field** that is continuously differentiable. Then $\mathbf{F}$ is **path independent on** U if and only if the Jacobian matrix $J\mathbf{F}(\mathbf{x})$ is symmetric for all $\mathbf{x} \in U$.

4. (Theorem 6.1.4 in the book) In $\mathbb{R}^2$ and $\mathbb{R}^3$ Theorem 6.1.5 in the book becomes, respectively,

   a) Let U be a **simply connected open set** in $\mathbb{R}^2$ and let $\mathbf{F} : U \to \mathbb{R}^2$ be a **vector field** that is continuously differentiable. Then $\mathbf{F}$ is **path independent on** U if and only if

   $$\frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y) = 0 \qquad \text{for every } (x, y) \in U,$$

   where F_1, F_2 are the components of $\mathbf{F}$.

   b) Let U be a **simply connected open set** in $\mathbb{R}^3$ and let $\mathbf{F} : U \to \mathbb{R}^3$ be a **vector field** that is continuously differentiable. Then $\mathbf{F}$ is **path independent on** U if and only if

   $$\text{curl } \mathbf{F} = 0 \qquad \text{in } U.$$

Remark: Path independence of the vector field $\mathbf{F}$ depends on the set U. For example, the vector field

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

is path independent on the set $\{(x, y) : |y| > 0 \text{ or } x > 0\}$ (see problem 30 in §6.1 of the book) but it **is not** path independent on $\{(x, y) : (x, y) \neq (0, 0)\}$ (see example 6.1.2 in the book).

2 Green's theorem.

1. (Theorem 6.2.1 in the book) Let $U \subset \mathbb{R}^2$ be a **simply connected open set**, and $\mathbf{F} : U \to \mathbb{R}^2$ be a smooth **vector field**. Let R be a region contained in U with piecewise smooth **counterclockwise-oriented boundary** ∂R. Then

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{x} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy,$$

   where F_1, F_2 are the components of $\mathbf{F}$.

2. In differential form notation, this is written

$$\oint_{\partial R} F_1\, dx + F_2\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$

Remark: Green's theorem relates **a line integral over a closed curve (∂R) with a double integral over the region bounded by the curve (R)** . Pay attention to the orientation of ∂R!

3 The divergence theorem.

(Theorem 6.3.1 in the book) Let $V \subset \mathbb{R}^3$ be an **open connected** set, and let $\mathbf{F} : V \to \mathbb{R}^3$ be a smooth **vector field**. For any solid region S contained in V whose boundary ∂S is piecewise smooth and **oriented with the outward normal**, we have

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n}\ d\sigma = \iiint_S \operatorname{div} \mathbf{F}\ dx\, dy\, dz.$$

Remark: The divergence theorem relates a **surface integral over the boundary of a solid region S with a triple integral over S**. Be careful with the orientation of ∂S!

4 Stokes' theorem.

1. (Theorem 6.4.1 in the book) Let $U \subset \mathbb{R}^3$ be an **open connected set**, and let $\mathbf{F} : U \to \mathbb{R}^3$ be a **vector field** that is continuously differentiable. Let M be any piecewise smooth, simple, **oriented** surface lying in U, and let ∂M be a **positively oriented boundary path**. Then

$$\oint_{\partial M} \mathbf{F} \cdot d\mathbf{x} = \iint_M \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\ d\sigma.$$

2. If M' is another surface with the same boundary curve ∂M as M and same orientations, then

$$\oint_{\partial M} \mathbf{F} \cdot d\mathbf{x} = \iint_{M'} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\ d\sigma,$$

   and, hence,

$$\iint_M \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\ d\sigma = \iint_{M'} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\ d\sigma.$$

Remark: Stokes' theorem relates a **line integral over a closed curve in $\mathbb{R}^3$ (∂M) with a surface integral over a surface (M) whose boundary is the curve**. Be careful with the orientations!